q-Conformal Invariant Equations and q-Plane Wave Solutions

V.K. Dobrev[1,2] and **S.T. Petrov**[2,3]

[1] School of Informatics,
University of Northumbria,
Newcastle upon Tyne NE1 8ST, UK
[2] Institute of Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria
(permanent address of both authors)
[3] Laboratoire de Physique des Matériaux,
Université Henri Poincaré Nancy-I
B.P. 239, 54506 Vandoeuvre les Nancy, Nancy, France

Abstract

We give new solutions of the quantum conformal deformations of the full Maxwell equations in terms of deformations of the plane wave. We study the compatibility of these solutions with the conservation of the current. We also start the study of quantum linear conformal (Weyl) gravity by writing the corresponding q-deformed equations.

Introduction

One of the purposes of quantum deformations is to provide an alternative of the regularization procedures of quantum field theory. Applied to Minkowski

1

space-time the quantum deformations approach is also an alternative to Connes' noncommutative geometry [1]. The first step in such an approach is to construct a noncommutative quantum deformation of Minkowski space-time. There are several possible such deformations, cf. [2, 3, 4, 5, 6]. We shall follow the deformation of [6] which is different from the others, the most important aspect being that it is related to a deformation of the conformal group.

The first problem to tackle in a noncommutative deformed setting is to study the q-deformed analogues of the conformally invariant equations. Here we continue the study of hierarchies of deformed equations derived in [6, 7, 8] with the use of quantum conformal symmetry. We give now a description of our setting starting from the simplest example.

It is well known that the d'Alembert equation

$$\Box\,\varphi(x) \;=\; 0\;, \qquad \Box \;=\; \partial^\mu\partial_\mu \;=\; (\partial_0)^2 - (\vec{\partial})^2\;, \qquad (0.1)$$

is conformally invariant, cf., e.g., [9]. Here φ is a scalar field of fixed conformal weight, $x \;=\; (x_0, x_1, x_2, x_3)$ denotes the Minkowski space-time coordinates. Not known was the fact that (0.1) may be interpreted as conditionally conformally invariant equation and thus may be rederived from a subsingular vector of a Verma module of the algebra $sl(4)$, the complexification of the conformal algebra $su(2,2)$ [7].

The same idea was used in [7] to derive a q-d'Alembert equation, namely, as arising from a subsingular vector of a Verma module of the quantum algebra $U_q(sl(4))$. The resulting equation is a q-difference equation and the solution spaces are built on the noncommutative q-Minkowski space-time of [6].

Besides the q-d'Alembert equation in [7] were derived a whole hierarchy of equations corresponding to the massless representations of the conformal group and parametrized by a nonnegative integer r [7]. The case $r = 0$ corresponds to the q-d'Alembert equation, while for each $r > 0$ there are two couples of equations involving fields of conjugated Lorentz representations of dimension $r + 1$. For instance, the case $r = 1$ corresponds to the massless Dirac equation, one couple of equations describing the neutrino, the other couple of equations describing the antineutrino, while the case $r = 2$ corresponds to the Maxwell equations.

The construction of solutions of the q-d'Alembert hierarchy was started in [10] with the q-d'Alembert equation. One of the solutions given was a

2

deformation of the plane wave as a formal power series in the noncommutative coordinates of q-Minkowski space-time and four-momenta. This q-plane wave has some properties analogous to the classical one but is not an exponent or q-exponent. Thus, it differs conceptually from the classical plane wave and may serve as a regularization of the latter. In the same sense it differs from the q-plane wave in the paper [11], which is not surprising, since there is used different q-Minkowski space-time (from [2, 3, 4] and different q-d'Alembert equation both based only on a (different) q-Lorentz algebra, and not on q-conformal (or $U_q(sl(4))$) symmetry as in our case. In fact, it is not clear whether the q-Lorentz algebra of [2, 3, 4] used in [11] is extendable to a q-conformal algebra.

For the equations labelled by $r > 0$ it turned out that one needs a second q-deformation of the plane wave in a conjugated basis [12]. The solutions of the hierarchy in terms of the two q-plane waves were given in [12] for $r = 1$ and in [13] for $r > 1$. Later these two q-plane waves were generalized and correspondingly more general solutions of the hierarchy were given in [14].

Another hierarchy derived in [6] is the Maxwell hierarchy. The two hierarchies have only one common member - the Maxwell equations - they are the lowest member of the Maxwell hierarchy and the $r = 2$ member of the massless hierarchy. The compatibility of the solutions of the free q-Maxwell equations with the q-potential equations was studied [15].

In the present paper we study the full q-Maxwell equations and the compatibility of their solutions with the conservation of the current. We give new solutions of the full q-Maxwell equations in two conjugated bases. The solutions of the homogeneous equations are also new (generalizing previously given solutions).

Another family contained in [8], but not explicated there, is related with the linear conformal (Weyl) gravity which we start to study in this paper. Namely, in the last Section we write down the quantum conformal deformations of the linear conformal (Weyl) gravity.

1 Preliminaries

First we introduce new Minkowski variables:

$$x_\pm \ \equiv \ x_0 \pm x_3 \ , \qquad v \ \equiv \ x_1 - ix_2 \ , \qquad \bar{v} \ \equiv \ x_1 + ix_2 \ , \qquad (1.1)$$

which, (unlike the x_μ), have definite group–theoretical interpretation as part of a six-dimensional coset of the conformal group $SU(2,2)$ (as explained in [6]). In terms of these variables, e.g., the d'Alembert equation (0.1) is:

$$\Box\,\varphi = (\partial_-\partial_+ \; - \; \partial_v\,\partial_{\bar{v}})\,\varphi \; = \; 0 \; . \tag{1.2}$$

In the q-deformed case we use the noncommutative q-Minkowski space-time of [6] which is given by the following commutation relations (with $\lambda \equiv q - q^{-1}$):

$$x_\pm v = q^{\pm 1} v x_\pm \, , \quad x_\pm\bar{v} = q^{\pm 1}\bar{v}x_\pm \, , \quad x_+x_- - x_-x_+ = \lambda v\bar{v} \, , \quad \bar{v}v = v\bar{v} \, , \quad (1.3)$$

with the deformation parameter being a phase: $|q| = 1$. Relations (1.3) are preserved by the anti-linear anti-involution ω :

$$\omega(x_\pm) \; = \; x_\pm \, , \quad \omega(v) \; = \; \bar{v} \, , \quad \omega(q) \; = \; \bar{q} \; = \; q^{-1} \, , \quad (\omega(\lambda) \; = \; -\lambda) \, . \tag{1.4}$$

The solution spaces consist of formal power series in the q-Minkowski coordinates (which we give in two conjugate bases):

$$\varphi \; = \; \sum_{j,n,\ell,m\in\mathbb{Z}_+} \mu_{jn\ell m}\,\varphi_{jn\ell m} \, , \qquad \varphi_{jn\ell m} \; = \; \hat{\varphi}_{jn\ell m},\;\tilde{\varphi}_{jn\ell m} \, , \quad (1.5)$$

$$\hat{\varphi}_{jn\ell m} \; = \; v^j\,x_-^n\,x_+^\ell\,\bar{v}^m \, , \tag{1.6}$$

$$\tilde{\varphi}_{jn\ell m} \; = \; \bar{v}^m\,x_+^\ell\,x_-^n\,v^j \; = \; \omega(\hat{\varphi}_{jn\ell m}) \, . \tag{1.7}$$

The solution spaces (1.5) are representation spaces of the quantum algebra $U_q(sl(4))$. For the latter we use the rational basis of Jimbo [16]. The action of $U_q(sl(4))$ on $\hat{\varphi}_{jn\ell m}$ was given in [17], and on $\tilde{\varphi}_{jn\ell m}$ in [12]. Because of the conjugation ω we are actually working with the conformal quantum algebra which is a deformation of $U(su(2,2))$.

Further we suppose that q is not a nontrivial root of unity.

In order to write our q-deformed equations in compact form it is necessary to introduce some additional operators. We first define the operators:

$$\hat{M}_\kappa^\pm\,\varphi \; = \; \sum_{j,n,\ell,m\in\mathbb{Z}_+}\mu_{jn\ell m}\,\hat{M}_\kappa^\pm\,\varphi_{jn\ell m} \, , \qquad \kappa = \pm, v, \bar{v} \, , \tag{1.8}$$

$$T_\kappa^\pm\,\varphi \; = \; \sum_{j,n,\ell,m\in\mathbb{Z}_+}\mu_{jn\ell m}\,T_\kappa^\pm\,\varphi_{jn\ell m} \, , \qquad \kappa = \pm, v, \bar{v} \, , \tag{1.9}$$

and $\hat{M}_+^\pm$, $\hat{M}_-^\pm$, $\hat{M}_v^\pm$, $\hat{M}_{\bar{v}}^\pm$, resp., acts on $\varphi_{jn\ell m}$ by changing by ± 1 the value of j, n, ℓ, m, resp., while $T_+^\pm$, $T_-^\pm$, $T_v^\pm$, $T_{\bar{v}}^\pm$, resp., acts on $\varphi_{jn\ell m}$

4

by multiplication by $q^{\pm j}, q^{\pm n}, q^{\pm \ell}, q^{\pm m}$, resp. We shall use also the 'logs' N_κ such that $T_\kappa = q^{N_\kappa}$. Now we can define the q-difference operators:

$$\hat{\mathcal{D}}_\kappa \, \varphi \;=\; \frac{1}{\lambda} \, \hat{M}_\kappa^{-1} \, \left(T_\kappa - T_\kappa^{-1} \right) \, \varphi \;=\; \frac{1}{\lambda} \, \hat{M}_\kappa^{-1} \, \left(q^{N_\kappa} - q^{-N_\kappa} \right) \, \varphi \,. \qquad (1.10)$$

Note that when $q \to 1$ then $\hat{\mathcal{D}}_\kappa \to \partial_k$. Using (1.8) and (1.10) the q-d'Alembert equation may be written as [7], [12], respectively,

$$\left(q \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, T_v \, T_{\bar{v}} \;-\; \hat{\mathcal{D}}_v \, \hat{\mathcal{D}}_{\bar{v}} \right) \, T_v \, T_- \, T_+ \, T_{\bar{v}} \, \hat{\varphi} \;=\; 0 \,, \qquad (1.11)$$

$$\left(\hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \;-\; q \, \hat{\mathcal{D}}_v \, \hat{\mathcal{D}}_{\bar{v}} \, T_v \, T_{\bar{v}} \right) \, T_- \, T_+ \, \tilde{\varphi} \;=\; 0 \,. \qquad (1.12)$$

Note that when $q \to 1$ both equations (1.11), (1.12) go to (1.2). Note that the operators in (1.8), (1.10), (1.11), (1.12) for different variables commute, i.e., we have passed to commuting variables. However, keeping the normal ordering it is straightforward to pass back to noncommuting variables.

Next we recall that the Maxwell's equations are part also of Maxwell's hierarchy of equations. The quantum conformal deformation of the equations of the hierarchy are [6]:

$$_q I_n^+ \; _q F_n^+ \;=\; _q J^n \,, \qquad _q I_n^- \; _q F_n^- \;=\; _q J^n \qquad (1.13)$$

where in the basis (1.6) the operators are:

$$\begin{aligned}
_q I_n^+ \;=\; & \tfrac{1}{2} \left(\left(q \, \hat{\mathcal{D}}_v \;+\; \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_+ \, (T_- \, T_v)^{-1} \, T_{\bar{v}} \right) T_- \, [n + 2 - N_z]_q \;-\; \right. \\
& \left. -\; q^{-n-2} \left(\hat{\mathcal{D}}_- \, T_- \;+\; q^{-1} \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_{\bar{v}} \;-\; \right. \right. \\
& \left. \left. -\; \lambda \, \hat{M}_v \, \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, T_{\bar{v}} \right) T_-^{-1} \, \hat{\mathcal{D}}_z \right) T_+ \, T_v \, T_z \, T_{\bar{z}}^{-1} \,, \qquad (1.14)
\end{aligned}$$

$$\begin{aligned}
_q I_n^- \;=\; & \tfrac{1}{2} \left(\hat{\mathcal{D}}_{\bar{v}} \;+\; q \, \hat{M}_z \, \hat{\mathcal{D}}_+ \, T_{\bar{v}} \, T_- \, T_v^{-1} \;-\; \right. \\
& \left. -\; q \lambda \, \hat{M}_v \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, T_{\bar{v}} \right) T_{\bar{v}} \, [n + 2 - N_{\bar{z}}]_q \;-\; \\
& -\; \tfrac{1}{2} \, q^{n+3} \left(\hat{\mathcal{D}}_- \;+\; q \, \hat{M}_z \, \hat{\mathcal{D}}_v \, T_- \right) \hat{\mathcal{D}}_{\bar{z}} \, T_- \, T_{\bar{v}} \,, \qquad (1.15)
\end{aligned}$$

while where in the basis (1.7) the operators are:

$$_q I_n^+ \;=\; \tfrac{1}{2} \, q \left(\hat{\mathcal{D}}_v \;+\; \hat{M}_{\bar{z}} \, \hat{\mathcal{D}}_+ \, T_- \, T_{\bar{v}}^{-1} \, T_v \right) T_v \, [n + 2 - N_z]_q \;-\;$$

$$- \tfrac{1}{2}\, q^{n+3} \left(\hat{\mathcal{D}}_- \ + \ \hat{M}_{\bar{z}}\, \hat{\mathcal{D}}_{\bar{v}}\, T_- \ + \right.$$

$$\left. + \ \lambda\, q^{-1} \hat{M}_v\, \hat{M}_{\bar{z}}\, \hat{\mathcal{D}}_-\, \hat{\mathcal{D}}_+\, T_{\bar{v}}^{-1}\, T_- \right) \hat{\mathcal{D}}_z\, T_-\, T_v \ , \qquad (1.16)$$

$$_q I_n^- \ = \ \tfrac{1}{2} \left(\left(\hat{\mathcal{D}}_{\bar{v}}\, T_{\bar{v}}\, T_- \ + \ \hat{M}_z\, \hat{\mathcal{D}}_+\, T_v \ + \right. \right.$$

$$+ \ q^{-1} \lambda\, \hat{M}_v\, \hat{\mathcal{D}}_-\, \hat{\mathcal{D}}_+\, T_- \left. \right) [n+2-N_{\bar{z}}]_q \ -$$

$$\left. - \ q^{-n-2} \left(\hat{\mathcal{D}}_- \ + \ \hat{M}_z\, \hat{\mathcal{D}}_v\, T_-^{-1} \right) \hat{\mathcal{D}}_{\bar{z}}\, T_{\bar{v}} \right) T_+\, T_{\bar{z}}\, T_z^{-1} \ . \qquad (1.17)$$

Note that for $q = 1$ (1.14),(1.15) coincide with (1.16),(1.17), respectively. Maxwell's equations $\partial^\mu F_{\mu\nu} = J_\nu$, $\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0$ are obtained from (1.13) for $n = 0$, $q = 1$, substituting the fixed helicity constituents $F^\pm$ by: $F^+ = z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+)$, $F^- = \bar{z}^2(F_1^- - iF_2^-) - 2\bar{z}F_3^- - (F_1^- + iF_2^-)$, $F_k^\pm = F_{k0} \pm \tfrac{i}{2}\varepsilon_{k\ell m}F_{\ell m} = E_k \pm iH_k$, $J^0 = \bar{z}z(J_0 + J_3) + z(J_1 + iJ_2) + \bar{z}(J_1 - iJ_2) + (J_0 - J_3)$, and then comparing the coefficients of the resulting first order polynomials in z and $\bar{z}$.

   We shall look for solutions of the full q-Maxwell's equations in terms of deformations of the plane wave. Let us first recall these deformations from [14]. The first deformation is given in the basis (1.6):

$$\widehat{\exp}_q(k, x) \ = \ \sum_{s=0}^{\infty} \frac{1}{[s]_q!}\, \hat{h}_s \ , \qquad (1.18)$$

$$[s]_q! \equiv [s]_q[s-1]_q \cdots [1]_q \ , \quad [0]_q! \equiv 1 \ , \quad [n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \ ,$$

$$\hat{h}_s \ = \ \beta^s \sum_{a,b,n \in \mathbb{Z}_+} \frac{(-1)^{s-a-b}\, q^{n(s-2a-2b+2n)\, +\, a(s-a-1)\, +\, b(-s+a+b+1)}\, q^{P_s(a,b)}}{\Gamma_q(a-n+1)\Gamma_q(b-n+1)\Gamma_q(s-a-b+n+1)[n]_q!} \ \times$$

$$\times \ k_v^{s-a-b+n} k_-^{b-n} k_+^{a-n} k_{\bar{v}}^n v^n x_-^{a-n} x_+^{b-n} \bar{v}^{s-a-b+n} \ , \qquad (1.19)$$

$$(\beta^s)^{-1} \ = \ \sum_{p=0}^{s} \frac{q^{(s-p)(p-1)+p}}{[p]_q!\, [s-p]_q!} \ ,$$

where the momentum components $(k_v, k_-, k_+, k_{\bar{v}})$ are supposed to be non-commutative between themselves (obeying the same rules (1.3) as the q-Minkowski coordinates), and commutative with the coordinates. Further, Γ_q is a q-deformation of the Γ-function, of which here we use only the properties:

$\Gamma_q(p) = [p-1]_q!$ for $p \in I\!N$, $1/\Gamma_q(p) = 0$ for $p \in Z\!\!\!Z_-$; $P_s(a,b)$ is a polynomial in a, b. Note that $(\hat{h}_s)|_{q=1} = (k\cdot x)^s$ and thus $(\widehat{\exp}_q(k,x))|_{q=1} = \exp(k\cdot x)$. This q-plane wave has some properties analogous to the classical one but is not an exponent or q-exponent, cf. [18]. This is enabled also by the fact (true also for $q = 1$) that solving the equations may be done in terms of the components $\hat{h}_s$. This deformation of the plane wave generalizes the original one from [10] to obtain which one sets $P_s(a,b) = 0$, in which case we shall use the notation f_s for the components from [10] since:

$$\left(\hat{h}_s\right)_{P_s(a,b)=0} = f_s . \tag{1.20}$$

Each $\hat{h}_s$ satisfies the q-d'Alembert equation (1.11) on the momentum q-cone:

$$\mathcal{L}_q^k \equiv k_-k_+ - q^{-1}k_v k_{\bar{v}} = k_+k_- - q k_v k_{\bar{v}} = 0 . \tag{1.21}$$

The second deformation is given in the basis (1.7):

$$\widetilde{\exp}_q(k,x) = \sum_{s=0}^{\infty} \frac{1}{[s]_q!} \, \tilde{h}_s , \tag{1.22}$$

$$\tilde{h}_s = \tilde{\beta}^s \sum_{a,b,n} \frac{(-1)^{s-a-b} \, q^{n(2a+2b-2n-s) \, + \, a(a-s-1) \, + \, b(s-a-b+1)} \, q^{Q_s(a,b)}}{\Gamma_q(a-n+1) \, \Gamma_q(b-n+1) \, \Gamma_q(s-a-b+n+1) \, [n]_q!} \times$$

$$\times \, k_{\bar{v}}^n k_+^{a-n} k_-^{b-n} k_v^{s-a-b+n} \, \bar{v}^{s-a-b+n} x_+^{b-n} x_-^{a-n} v^n , \tag{1.23}$$

$$\left(\tilde{\beta}^s\right)^{-1} = \sum_{p=0}^{s} \frac{q^{(p-s)(p-1)+p}}{[p]_q! \, [s-p]_q!} ,$$

where $Q_s(a,b)$ are arbitrary polynomials. If the latter are zero then $\widetilde{\exp}_q(k,x)$ becomes the q-plane wave deformation found in [12]. The $\tilde{h}_s$ have the same properties as the $\hat{h}_s$ but the conjugated basis is used; in particular, they satisfy the q-d'Alembert equation (1.12) on the momentum q-cone (1.21).

2 Solutions of the q-Maxwell equations

First we shall use the basis (1.6). The solutions of (1.13) for $n = 0$ in the homogeneous case ($J = 0$) are:

$$\hat{F}^{h\pm} \doteq \left({}_qF_0^{\pm}\right)_{J=0} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \, \hat{F}_{ms}^{h\pm}(k) \, f_s , \tag{2.1}$$

7

$$\hat{F}_{ms}^{h+}(k) = \sum_{i=0}^{m} \Big(\sum_{j=0}^{m-i} \hat{p}_{ij}^{ms1} \, k_v^i k_-^{m-i-j} k_{\bar{v}}^j \, (k_v - q^{s+6} z k_-)(k_v - q^{s+3} z k_-) \; +$$

$$+ \; \hat{p}_i^{ms2} \, k_v^i k_{\bar{v}}^{m-i} \, (k_v - q^{s+6} z k_-)(k_+ - q^{s+3} z k_{\bar{v}}) \; +$$

$$+ \; \sum_{j=0}^{m-i} \hat{p}_{ij}^{ms3} \, k_v^i k_+^{m-i-j} k_{\bar{v}}^j \, (k_+ - q^{s+6} z k_{\bar{v}})(k_+ - q^{s+3} z k_{\bar{v}}) \Big) \; , (2.2)$$

$$\hat{F}_{ms}^{h-}(k) = \sum_{i=0}^{m} \Big(\sum_{j=0}^{m-i} \hat{r}_{ij}^{ms1} \, k_v^i k_-^{m-i-j} k_{\bar{v}}^j \, (k_{\bar{v}} - q^{-1} \bar{z} k_-)(k_{\bar{v}} - \bar{z} k_-) \; +$$

$$+ \; \hat{r}_i^{ms2} \, k_v^i k_{\bar{v}}^{m-i} \, (k_+ - q^{-1} \bar{z} k_v)(k_{\bar{v}} - \bar{z} k_-) \; +$$

$$+ \; \sum_{j=0}^{m-i} \hat{r}_{ij}^{ms3} \, k_v^i k_+^{m-i-j} k_{\bar{v}}^j \, (k_+ - q^{-1} \bar{z} k_v)(k_+ - \bar{z} k_v) \Big) \; , \qquad (2.3)$$

where $\hat{p}_{i(j)}^{msa}, \hat{r}_{i(j)}^{msa}$ are independent constants. The check that these are solutions is done for commutative Minkowski coordinates and noncommutative momenta on the q-cone. The terms with $m = 0$ of the solutions (2.1), (2.2), (2.3), were obtained earlier [13] (later they were generalized using more general q-plane waves [14]). The solution (2.3) can be written in terms of the deformed plane wave if we suppose that the $\hat{r}_{i(j)}^{msa}$ for different s coincide: $\hat{r}_{i(j)}^{msa} = \hat{r}_{i(j)}^{ma}$. Then we have:

$$\hat{F}^{h-} = \sum_{m=0}^{\infty} \hat{F}_m^{h-}(k) \exp_q(k, x) \; , \quad \hat{F}_m^{h-}(k) = \hat{F}_{ms}^{h-}(k) \; . \qquad (2.4)$$

In the inhomogeneous case the solutions of (1.13) for $n = 0$ are:

$$_qJ^0 = \bar{z}z\hat{J}_+ + z\hat{J}_v + \bar{z}\hat{J}_{\bar{v}} + \hat{J}_- \; , \qquad (2.5)$$

$$\hat{J}_\kappa = \sum_{m,s=o}^{\infty} \frac{1}{[s]_q!} \, \hat{J}_\kappa^{ms}(k) \, f_{s-1} \; , \quad \kappa = \pm, v, \bar{v} \; , \qquad (2.6)$$

$$\hat{J}_+^{ms}(k) = -\hat{K}_m^s(k) \, k_- \; , \qquad (2.7)$$

$$\hat{J}_-^{ms}(k) = -q^{-s-2} \hat{K}_m^s(k) \, k_+ \; ,$$

$$\hat{J}_v^{ms}(k) = \hat{K}_m^s(k) \, k_{\bar{v}} \; ,$$

$$\hat{J}_{\bar{v}}^{ms}(k) = q^{-s-2} \hat{K}_m^s(k) \, k_v \; ,$$

$$\hat{K}_m^s(k) \doteq \hat{\gamma}_v^s k_v^{m+1} + \hat{\gamma}_-^s k_-^{m+1} + \hat{\gamma}_+^s k_+^{m+1} + \hat{\gamma}_{\bar{v}}^s k_{\bar{v}}^{m+1} \; ;$$

8

$$_q F_0^{\pm} \;=\; \hat{F}^{\pm} \;+\; \hat{F}^{h\pm} \;, \tag{2.8}$$

$$\hat{F}^{\pm} \;=\; \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \, \hat{F}_{ms}^{\pm}(k) \, f_s \;, \tag{2.9}$$

$$\hat{F}_{ms}^{+}(k) \;=\; 2 d_s q^{-s} ((q^{-s-5} \hat{\gamma}_-^s k_-^m + z \hat{\gamma}_v^s k_v^m)(k_v - q^{s+3} z k_-) \;+$$
$$+\; (q^{-s-5} \hat{\gamma}_{\bar{v}}^s k_{\bar{v}}^m + z \hat{\gamma}_+^s k_+^m)(k_+ - q^{s+3} z k_{\bar{v}})) \;,$$

$$\hat{F}_{ms}^{-}(k) \;=\; 2 d_s q^{-2s-2} ((\hat{\gamma}_-^s k_-^m + q^{-2} \bar{z} \hat{\gamma}_{\bar{v}}^s k_{\bar{v}}^m)(k_{\bar{v}} - \bar{z} k_-) \;+$$
$$+\; (\hat{\gamma}_v^s k_v^m + q^{-2} \bar{z} \hat{\gamma}_+^s k_+^m)(k_+ - \bar{z} k_v)) \;,$$

where $d_s = \beta^s / \beta^{s+1}$. As in the homogeneous case we can not make $\hat{F}_{ms}^{+}(k)$ independent of s. We can make $\hat{F}_{ms}^{-}(k)$ independent of s by choosing $\hat{\gamma}_\kappa^s \sim q^{2s} d_s^{-1}$, but we can not make $\hat{J}_\kappa^{ms}(k)$ independent of s.

Since we work with the full Maxwell equations we have also to check the q-deformation of the current conservation $\partial^\nu J_\nu = 0$:

$$I_{13} \, J \;=\; 0 \;, \tag{2.10}$$

$$I_{13} \;=\; q^3 \, [N_z - 1]_q \, T_z \, \hat{\mathcal{D}}_{\bar{z}} \, \hat{\mathcal{D}}_v \, T_v \, T_- \, T_+ \;+\; q \, \hat{\mathcal{D}}_z \, T_z \, \hat{\mathcal{D}}_{\bar{z}} \, \hat{\mathcal{D}}_- \, T_v \, T_+ \;+$$
$$+\; q \, [N_z - 1]_q \, T_z \, [N_{\bar{z}} - 1]_q \, \hat{\mathcal{D}}_+ \, T_+ \, T_{\bar{v}} \;+$$
$$+\; q^{-1} \, [N_{\bar{z}} - 1]_q \, \hat{\mathcal{D}}_z \, T_z \, \hat{\mathcal{D}}_{\bar{v}} \, T_v \, T_-^{-1} \, T_+ \;-$$
$$-\; \lambda \, \hat{M}_v \, [N_{\bar{z}} - 1]_q \, \hat{\mathcal{D}}_z \, T_z \, \hat{\mathcal{D}}_- \, \hat{\mathcal{D}}_+ \, T_v \, T_-^{-1} \, T_+ \, T_{\bar{v}} \tag{2.11}$$

Substituting (2.5,2.6) in the above we get:

$$q \, J_+^s(k) \, k_+ \;+\; J_v^s(k) \, k_v \;+\; q^{s+2} \, J_{\bar{v}}^s \, k_{\bar{v}} \;+\; q^{s+1} \, J_-^s(k) \, k_- = 0 \tag{2.12}$$

The latter is fulfilled by the explicit expressions in (2.7), but we should note that these expressions fulfil also the following splittings of (2.12) :

$$q \, J_+^s(k) \, k_+ \;+\; J_v^s(k) \, k_v \;=\; 0 \;, \quad q \, J_{\bar{v}}^s(k) \, k_{\bar{v}} \;+\; J_-^s(k) \, k_- \;=\; 0 \;, \tag{2.13}$$
$$J_+^s(k) \, k_+ \;+\; q^{s+1} \, J_{\bar{v}}^s(k) \, k_{\bar{v}} \;=\; 0 \;, \quad J_v^s(k) \, k_v \;+\; q^{s+1} \, J_-^s(k) \, k_- \;=\; 0 \;.$$

Furthermore the expressions from (2.7) fulfil also:

$$q \, J_+^s(k) \, k_{\bar{v}} \;+\; J_v^s(k) \, k_- \;=\; 0 \;, \quad q \, J_{\bar{v}}^s(k) \, k_+ \;+\; J_-^s(k) \, k_v \;=\; 0 \;, \tag{2.14}$$
$$J_+^s(k) \, k_v \;+\; q^{s+1} \, J_{\bar{v}}^s(k) \, k_- \;=\; 0 \;, \quad J_v^s(k) \, k_+ \;+\; q^{s+1} \, J_-^s(k) \, k_{\bar{v}} \;=\; 0 \;.$$

9

Now we shall use the basis (1.7). Then solutions of (1.13) for $n = 0$ in the homogeneous case $(J = 0)$ are:

$$\tilde{F}^{h\pm} \doteq \left({}_qF_0^\pm \right)_{J=0} = \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!} \tilde{F}_{ms}^{h\pm}(k)\, \tilde{h}_s \;, \qquad (2.15)$$

$$\tilde{F}_{ms}^{h+}(k) = \sum_{i=0}^{m} \Big(\sum_{j=0}^{m-i} \tilde{p}_{ij}^{ms1}\, k_{\bar{v}}^i k_-^{m-i-j} k_v^j\, (k_v - zk_-)(k_v - qzk_-) \; +$$

$$+\; \tilde{p}_i^{ms2}\, k_{\bar{v}}^i k_v^{m-i}\, (k_+ - zk_{\bar{v}})(k_v - qzk_-) \; +$$

$$+\; \sum_{j=0}^{m-i} \tilde{p}_{ij}^{ms3}\, k_{\bar{v}}^i k_+^{m-i-j} k_v^j\, (k_+ - zk_{\bar{v}})(k_+ - qzk_{\bar{v}}) \, \Big) \;, \qquad (2.16)$$

$$\tilde{F}_{ms}^{h-}(k) = \sum_{i=0}^{m} \Big(\sum_{j=0}^{m-i} \tilde{r}_{ij}^{ms1}\, k_{\bar{v}}^i k_-^{m-i-j} k_v^j\, (k_{\bar{v}} - q^{s+1}\bar{z}k_-)(k_{\bar{v}} - q^{s+2}\bar{z}k_-) \; +$$

$$+\; \tilde{r}_i^{ms2}\, k_v^i k_{\bar{v}}^{m-i}\, (k_{\bar{v}} - q^{s+1}\bar{z}k_-)(k_+ - q^{s+2}\bar{z}k_v) \; +$$

$$+\; \sum_{j=0}^{m-i} \tilde{r}_{ij}^{ms3}\, k_v^i k_+^{m-i-j} k_{\bar{v}}^j\, (k_+ - q^{s+1}\bar{z}k_v)(k_+ - q^{s+2}\bar{z}k_v) \, \Big) (2.17)$$

where $\tilde{p}_{i(j)}^{msa}, \tilde{r}_{i(j)}^{msa}$ are independent constants, $Q_s(a,b) = 0$ in $\tilde{h}_s$. The terms with $m = 0$ of the solutions (2.15), (2.16), (2.17) were obtained earlier in [13] (and using the generalized q-plane wave in [14]). The solution (2.16) can be written in terms of the deformed plane wave if we suppose that the $\tilde{p}_{i(j)}^{msa}$ for different s coincide: $\tilde{p}_{i(j)}^{msa} = \tilde{p}_{i(j)}^{ma}$. Then we have:

$$\tilde{F}^{h+} = \sum_{m=0}^{\infty} \tilde{F}_m^{h+}(k)\, \widetilde{\exp}_q(k,x) \;, \quad \tilde{F}_m^{h+}(k) = \tilde{F}_{ms}^{h+}(k) \;. \qquad (2.18)$$

In the inhomogeneous case the solutions of (1.13) for $n = 0$ are:

$$_qJ^0 = \bar{z}z\tilde{J}_+ + z\tilde{J}_v + \bar{z}\tilde{J}_{\bar{v}} + \tilde{J}_- \;, \qquad (2.19)$$

$$\tilde{J}_\kappa = \sum_{m,s=o}^{\infty} \frac{1}{[s]_q!} \tilde{J}_\kappa^{ms}(k)\, \tilde{h}_{s-1} \;, \quad \kappa = \pm, v, \bar{v} \;, \qquad (2.20)$$

$$\tilde{J}_+^{ms}(k) = -q^{s+1}\tilde{K}_m^s(k)\, k_- \;, \qquad (2.21)$$

10

$$\tilde{J}_{-}^{ms}(k) \quad = \quad -q^{-1}\tilde{K}_m^s(k)\,k_+ \ ,$$

$$\tilde{J}_{v}^{ms}(k) \quad = \quad \tilde{K}_m^s(k)\,k_{\bar{v}} \ ,$$

$$\tilde{J}_{\bar{v}}^{ms}(k) \quad = \quad q^s\tilde{K}_m^s(k)\,k_v \ ,$$

$$\tilde{K}_m^s(k) \quad \doteq \quad \tilde{\gamma}_v^s k_v^{m+1} + \tilde{\gamma}_-^s k_-^{m+1} + \tilde{\gamma}_+^s k_+^{m+1} + \tilde{\gamma}_{\bar{v}}^s k_{\bar{v}}^{m+1} \ ,$$

$$_qF_0^\pm \quad = \quad \tilde{F}^\pm \ + \ \tilde{F}^{h\pm} \ , \tag{2.22}$$

$$\tilde{F}^\pm \quad = \quad \sum_{m,s=0}^{\infty} \frac{1}{[s]_q!}\,\tilde{F}_{ms}^\pm(k)\,\tilde{h}_s \ , \tag{2.23}$$

$$\tilde{F}_{ms}^+(k) \quad = \quad 2\tilde{d}_s q^{s-2}((\tilde{\gamma}_-^s k_-^m + q^{-1}z\tilde{\gamma}_v^s k_v^m)(k_v - qzk_-) \ +$$

$$+ \ (\tilde{\gamma}_{\bar{v}}^s k_{\bar{v}}^m + q^{-1}z\tilde{\gamma}_+^s k_+^m)(k_+ - qzk_{\bar{v}})) \ ,$$

$$\tilde{F}_{ms}^-(k) \quad = \quad 2\tilde{d}_s((q^{-s-3}\tilde{\gamma}_-^s k_-^m + q\bar{z}\tilde{\gamma}_{\bar{v}}^s k_{\bar{v}}^m)(k_{\bar{v}} - q^{s+2}\bar{z}k_-) \ +$$

$$+ \ (q^{-s-3}\tilde{\gamma}_v^s k_v^m + q\bar{z}\tilde{\gamma}_+^s k_+^m)(k_+ - q^{s+2}\bar{z}k_v)) \ ,$$

where $\tilde{d}_s = \tilde{\beta}^s/\tilde{\beta}^{s+1}$, $Q_s(a,b) = 0$ in $\tilde{h}_s$. We can not make $\tilde{F}_{ms}^-(k)$ or $\tilde{J}_\kappa^{ms}(k)$ independent of s. We can make $\tilde{F}_{ms}^+(k)$ independent of s by choosing $\tilde{\gamma}_\kappa^s \sim q^{-s}\tilde{d}_s^{-1}$.

Also here we shall check whether the q-deformation of the current conservation (2.10) is fulfilled. The analog of (2.11) in the basis (1.7) is:

$$I_{13} \quad = \quad [N_z - 1]_q\,\hat{\mathcal{D}}_{\bar{z}}\,T_{\bar{z}}\,\hat{\mathcal{D}}_v\,T_{\bar{v}}\,T_+\,T_-^{-1} \ + \ q\,\hat{\mathcal{D}}_{\bar{z}}\,T_{\bar{z}}\,\hat{\mathcal{D}}_z\,\hat{\mathcal{D}}_-\,T_{\bar{v}}\,T_+ \ +$$

$$+ \ q\,[N_{\bar{z}} - 1]_q\,T_{\bar{z}}\,[N_z - 1]_q\,\hat{\mathcal{D}}_+\,T_+\,T_v \ +$$

$$+ \ q^2\,[N_{\bar{z}} - 1]_q\,\hat{\mathcal{D}}_z\,T_{\bar{z}}\,\hat{\mathcal{D}}_{\bar{v}}\,T_{\bar{v}}\,T_-\,T_+ \ -$$

$$- \ \lambda\,q\,\hat{M}_v\,[N_{\bar{z}} - 1]_q\,\hat{\mathcal{D}}_z\,T_{\bar{z}}\,\hat{\mathcal{D}}_-\,\hat{\mathcal{D}}_+\,T_-\,T_+ \tag{2.24}$$

Then the analog of (2.12) is:

$$J_+^s(k)\,k_+ \ + \ q^s\,J_v^s(k)\,k_v \ + \ J_{\bar{v}}^s\,k_{\bar{v}} \ + \ q^s\,J_-^s(k)\,k_- = 0 \tag{2.25}$$

The latter is fulfilled by the explicit expressions in (2.21), but we should note that these expressions fulfil also the following splittings of (2.25) :

$$J_+^s(k)\,k_+ \ + \ q^s\,J_v^s(k)\,k_v \ = \ 0 \ , \quad J_{\bar{v}}^s(k)\,k_{\bar{v}} \ + \ q^s\,J_-^s(k)\,k_- \ = \ 0 \ ,$$

$$J_+^s(k)\,k_+ \ + \ J_{\bar{v}}^s(k)\,k_{\bar{v}} \ = \ 0 \ , \quad J_v^s(k)\,k_v \ + \ J_-^s(k)\,k_- \ = \ 0 \ . \tag{2.26}$$

11

Furthermore the expressions from (2.21) fulfil also:

$$J_+^s(k)\,k_{\bar v}\;+\;q^s\,J_v^s(k)\,k_-\;=\;0\;,\quad J_{\bar v}^s(k)\,k_+\;+\;q^s\,J_-^s(k)\,k_v\;=\;0\;,$$
$$J_+^s(k)\,k_v\;+\;J_{\bar v}^s(k)\,k_-\;=\;0\;,\quad J_v^s(k)\,k_+\;+\;J_-^s(k)\,k_{\bar v}\;=\;0\;. \tag{2.27}$$

Summarizing, we have given new solutions of the full q-Maxwell equations in two conjugated bases (1.6) and (1.7). The solutions of the homogeneous equations are also new (the old solutions are special cases). We see that the roles of the solutions F^+ and F^- are exchanged in the two conjugated bases. We note also that the currents components are different : $\hat J_\kappa^{ms}\;\neq\;\tilde J_\kappa^{ms}$ (for $q \neq 1$, $\kappa \neq v$), and in both cases they can not be made independent of s. Thus, there is no advantage of choosing either of the bases (1.6) or (1.7). It may be also possible to use both in a Connes-Lott type model [19].

3 Linear conformal gravity

We consider now the quantum group analogs of linear conformal gravity following the approach of [8]. We start with the $q = 1$ situation and we first write the Weyl gravity equations in an indexless formulation, trading the indices for two conjugate variables $z, \bar z$, just as for the Maxwell equations.

Weyl gravity is governed by the Weyl tensor:

$$C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} - \tfrac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\nu\tau}R_{\mu\sigma} - g_{\mu\tau}R_{\nu\sigma} - g_{\nu\sigma}R_{\mu\tau}) + \tfrac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R\;, \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor. Linear conformal gravity is obtained when the metric tensor is written as: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Minkowski metric, $h_{\mu\nu}$ are small so that all quadratic and higher order terms are neglected. In particular: $R_{\mu\nu\sigma\tau} = \tfrac{1}{2}(\partial_\mu\partial_\tau h_{\nu\sigma} + \partial_\nu\partial_\sigma h_{\mu\tau} - \partial_\mu\partial_\sigma h_{\nu\tau} - \partial_\nu\partial_\tau h_{\mu\sigma})$. The equations of linear conformal gravity are:

$$\partial^\nu\partial^\tau C_{\mu\nu\sigma\tau} = T_{\mu\sigma}\;, \tag{3.2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. From the symmetry properties of the Weyl tensor it follows that it has ten independent components. These may be chosen as follows (introducing notation for future use):

$$C_0 = C_{0123}\;,\quad C_1 = C_{2121}\;,\quad C_2 = C_{0202}\;,\quad C_3 = C_{3012}\;,$$
$$C_4 = C_{2021}\;,\quad C_5 = C_{1012}\;,\quad C_6 = C_{2023}\;,$$
$$C_7 = C_{3132}\;,\quad C_8 = C_{2123}\;,\quad C_9 = C_{1213}\;. \tag{3.3}$$

Furthermore, the Weyl tensor transforms as the direct sum of two conjugate Lorentz irreps, which we shall denote as $C^{\pm}$. The tensors $T_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric and traceless with nine independent components.

In order to be more precise we recall that the physically relevant representations T^{χ} of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1 + 1)(n_2 + 1)$), and d is the conformal dimension (or energy). (In the literature these Lorentz representations are labelled also by $(j_1, j_2) = (n_1/2, n_2/2)$.) The Weyl tensor transforms as the direct sum:

$$\chi^+ \oplus \chi^-$$

$$\chi^+ = [4, 0; 2] \ , \quad \chi^- = [0, 4; 2] \ , \tag{3.4}$$

while the energy-momentum tensor and the metric transform as:

$$\chi_T = [2, 2; 4] \ , \qquad \chi_h = [2, 2; 0] \ , \tag{3.5}$$

as anticipated. Indeed, $(n_1, n_2) = (2, 2)$ is the nine-dimensional Lorentz representation, (carried by $T_{\mu\nu}$ or $h_{\mu\nu}$), and $(n_1, n_2) = (4, 0), (0, 4)$ are the two conjugate five-dimensional Lorentz representations, (carried by $C^{\pm}$), while the conformal dimensions are the canonical dimensions of a energy-momentum tensor ($d = 4$), of the metric ($d = 0$), and of the Weyl tensor ($d = 2$). (For comparison, note that the Maxwell components F^+, F^-, used in the previous sections, have signatures: $[2, 0; 2]$, $[0, 2; 2]$, resp., while the current J has signature $[1, 1; 3]$.) Further, we shall use again the fact that a Lorentz irrep (spin-tensor) with signature (n_1, n_2) may be represented by a polynomial $G(z, \bar{z})$ in $z, \bar{z}$ of order n_1, n_2, resp. More explicitly, for the irreps mentioned above we use:

$$
\begin{aligned}
C^+(z) &= z^4 C_4^+ + z^3 C_3^+ + z^2 C_2^+ + z C_1^+ + C_0^+ \ , & \text{(3.6a)} \\
C^-(\bar{z}) &= \bar{z}^4 C_4^- + \bar{z}^3 C_3^- + \bar{z}^2 C_2^- + \bar{z} C_1^- + C_0^- \ , & \text{(3.6b)} \\
T(z, \bar{z}) &= z^2 \bar{z}^2 T_{22}' + z^2 \bar{z} T_{21}' + z^2 T_{20}' + \\
& \quad + z \bar{z}^2 T_{12}' + z \bar{z} T_{11}' + z T_{10}' + \\
& \quad + \bar{z}^2 T_{02}' + \bar{z} T_{01}' + T_{00}' \ , & \text{(3.6c)} \\
h(z, \bar{z}) &= z^2 \bar{z}^2 h_{22}' + z^2 \bar{z} h_{21}' + z^2 h_{20}' + \\
& \quad + z \bar{z}^2 h_{12}' + z \bar{z} h_{11}' + z h_{10}' +
\end{aligned}
$$

13

$$+\bar{z}^2 h'_{02} + \bar{z} h'_{01} + h'_{00} \ , \tag{3.6d}$$

where the indices on the RHS are not Lorentz-covariance indices, they just indicate the powers of $z, \bar{z}$. The components $C_k^{\pm}$ are given in terms of the Weyl tensor components as follows:

$$
\begin{aligned}
C_0^+ &= C_2 - \tfrac{1}{2}C_1 - C_6 + i(C_0 + \tfrac{1}{2}C_3 + C_7) \\
C_1^+ &= 2(C_4 - C_8 + i(C_9 - C_5)) \\
C_2^+ &= 3(C_1 - iC_3) \\
C_3^+ &= 8(C_4 + C_8 + i(C_9 + C_5)) \\
C_4^+ &= C_2 - \tfrac{1}{2}C_1 + C_6 + i(C_0 + \tfrac{1}{2}C_3 - C_7) \\
C_0^- &= C_2 - \frac{C_1}{2} - C_6 - i(C_0 + \tfrac{1}{2}C_3 + C_7) \\
C_1^- &= 2(C_4 - C_8 - i(C_9 - C_5)) \\
C_2^- &= 3(C_1 + iC_3) \\
C_3^- &= 2(C_4 + C_8 - i(C_9 + C_5)) \\
C_4^- &= C_2 - \tfrac{1}{2}C_1 + C_6 - i(C_0 + \tfrac{1}{2}C_3 - C_7)
\end{aligned} \tag{3.7}
$$

while the components T'_{ij} are given in terms of $T_{\mu\nu}$ as follows:

$$
\begin{aligned}
T'_{22} &= T_{00} + 2T_{03} + T_{33} \\
T'_{11} &= T_{00} - T_{33} \\
T'_{00} &= T_{00} - 2T_{03} + T_{33} \\
T'_{21} &= T_{01} + iT_{02} + T_{13} + iT_{23} \\
T'_{12} &= T_{01} - iT_{02} + T_{13} - iT_{23} \\
T'_{10} &= T_{01} + iT_{02} - T_{13} - iT_{23} \\
T'_{01} &= T_{01} - iT_{02} - T_{13} + iT_{23} \\
T'_{20} &= T_{11} + 2iT_{12} - T_{22} \\
T'_{02} &= T_{11} - 2iT_{12} - T_{22}
\end{aligned} \tag{3.8}
$$

and similarly for h'_{ij} in terms of $h_{\mu\nu}$.

In these terms all linear conformal (Weyl) gravity equations (3.2) may be written in compact form as the following pair of equations:

$$
\begin{aligned}
\tilde{I}^+ \ C^+(z) &= \ T(z, \bar{z}) \ , \tag{3.9a} \\
\tilde{I}^- \ C^-(\bar{z}) &= \ T(z, \bar{z}) \ , \tag{3.9b}
\end{aligned}
$$

where the operators $I^{\pm}$ are given as follows:

$$\tilde{I}^+ \ = \ \left(z^2 \bar{z}^2 \partial_+^2 + z^2 \partial_v^2 + \bar{z}^2 \partial_{\bar{v}}^2 + \partial_-^2 + \right.$$

14

$$
+2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + 2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) +
$$
$$
+2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_-\Big)\partial_z^2 -
$$
$$
-6\Big(z\bar{z}^2\partial_+^2 + z\partial_v^2 + 2z\bar{z}\partial_v\partial_+ + \bar{z}^2\partial_+\partial_{\bar{v}} +
$$
$$
+\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_v\partial_-\Big)\partial_z +
$$
$$
12\Big(\bar{z}^2\partial_+^2 + \partial_v^2 + 2\bar{z}\partial_v\partial_+\Big) \,, \tag{3.10a}
$$
$$
\tilde{I}^- = \Big(z^2\bar{z}^2\partial_+^2 + z^2\partial_v^2 + \bar{z}^2\partial_{\bar{v}}^2 + \partial_-^2 +
$$
$$
+2z^2\bar{z}\partial_v\partial_+ + 2z\bar{z}^2\partial_+\partial_{\bar{v}} + 2z\bar{z}(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) +
$$
$$
+2\bar{z}\partial_-\partial_{\bar{v}} + 2z\partial_v\partial_-\Big)\partial_{\bar{z}}^2 -
$$
$$
-6\Big(z^2\bar{z}\partial_+^2 + \bar{z}\partial_{\bar{v}}^2 + 2z\bar{z}\partial_+\partial_{\bar{v}} + z^2\partial_v\partial_+ +
$$
$$
+z(\partial_-\partial_+ + \partial_v\partial_{\bar{v}}) + \partial_-\partial_{\bar{v}}\Big)\partial_{\bar{z}} +
$$
$$
12\Big(z^2\partial_+^2 + \partial_{\bar{v}}^2 + 2z\partial_+\partial_{\bar{v}}\Big) \,. \tag{3.10b}
$$

To make more transparent the origin of these expressions and in the same time to derive the quantum group deformation of (3.9), (3.10) we first introduce the following parameter-dependent operators:

$$
\tilde{I}^+(n) = \tfrac{1}{2}\Big(n(n-1)I_1^2 I_2^2 - 2(n^2-1)I_1 I_2^2 I_1 + n(n+1)I_2^2 I_1^2\Big) \tag{3.11a}
$$
$$
\tilde{I}^-(n) = \tfrac{1}{2}\Big(n(n-1)I_3^2 I_2^2 - 2(n^2-1)I_3 I_2^2 I_3 + n(n+1)I_2^2 I_3^2 \,, \tag{3.11b}
$$

where

$$
I_1 \equiv \partial_z \,, \quad I_2 \equiv \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_- \,, \quad I_3 \equiv \partial_{\bar{z}} \,. \tag{3.12}
$$

It is easy to check that we have the following relation:

$$
\tilde{I}^\pm = \tilde{I}^\pm(4) \,. \tag{3.13}
$$

We note in passing that group-theoretically the operators I_a correspond to the three simple roots of the root system of $sl(4)$, while the operators $I_n^\pm$ correspond to the two non-simple non-highest roots [20].

This is the form that is immediately generalizable to the q-deformed case. Using results from [8] we have:

$$
_q\tilde{I}^+(n) = \tfrac{1}{2}\Big([n]_q [n-1]_q {}_qI_1^2 {}_qI_2^2 - [2]_q [n-1]_q [n+1]_q {}_qI_1 {}_qI_2^2 {}_qI_1 +
$$
$$
+[n]_q [n+1]_q {}_qI_2^2 {}_qI_1^2\Big) \,, \tag{3.14a}
$$
$$
_q\tilde{I}^-(n) = \tfrac{1}{2}\Big([n]_q [n-1]_q {}_qI_3^2 {}_qI_2^2 - [2]_q [n-1]_q [n+1]_q {}_qI_3 {}_qI_2^2 {}_qI_3 +
$$

15

$$+[n]_q\,[n+1]_q\,{}_qI_2^2\,{}_qI_3^2\Big)\ , \tag{3.14b}$$

where the q-deformed versions ${}_qI_a$ of (3.12) in the basis (1.6) are:

$$
\begin{aligned}
{}_qI_1 &= \hat{\mathcal{D}}_z T_z T_v T_+ (T_- T_{\bar v})^{-1} && \text{(3.15a)}\\
{}_qI_2 &= (q\hat{M}_z\hat{\mathcal{D}}_v T_-^2 + \hat{M}_z\hat{M}_{\bar z}\hat{\mathcal{D}}_+ T_- T_{\bar v} T_v^{-1} + \hat{\mathcal{D}}_- T_- + \\
&\quad + q^{-1}\hat{M}_{\bar z}\hat{\mathcal{D}}_{\bar v} - \lambda\hat{M}_v\hat{M}_{\bar z}\hat{\mathcal{D}}_-\hat{\mathcal{D}}_+ T_{\bar v})\, T_{\bar v} T_{\bar z}^{-1} && \text{(3.15b)}\\
{}_qI_3 &= \hat{\mathcal{D}}_{\bar z} T_{\bar z}\ . && \text{(3.15c)}
\end{aligned}
$$

Then the q-Weyl equations are:

$$
\begin{aligned}
{}_q\tilde{I}^+(4)\ C^+(z) &= T(z,\bar z)\ , && \text{(3.16a)}\\
{}_q\tilde{I}^-(4)\ C^-(\bar z) &= T(z,\bar z)\ . && \text{(3.16b)}
\end{aligned}
$$

(For comparison, note that for the derivation of the q-Maxwell operators (1.13) were used the following expressions: ${}_qI_n^+ = \frac{1}{2}([n+2]_q\,{}_qI_1\,{}_qI_2 - [n+3]_q\,{}_qI_2\,{}_qI_1)$, ${}_qI_n^- = \frac{1}{2}([n+2]_q\,{}_qI_3\,{}_qI_2 - [n+3]_q\,{}_qI_2\,{}_qI_3)$.)

Finally, we write down the pair of equations which give the Weyl tensor components in terms of the metric tensor:

$$
\begin{aligned}
{}_q\tilde{I}^+(2)\ h(z,\bar z) &= C^+(z)\ , && \text{(3.17a)}\\
{}_q\tilde{I}^-(2)\ h(z,\bar z) &= C^-(\bar z)\ . && \text{(3.17b)}
\end{aligned}
$$

We stress the advantage of the indexless formalism due to which two different pairs of equations, (3.16), (3.17), may be written using the same parameter-dependent operator expressions by just specializing the values of the parameter.

Acknowledgments

References

[1] A. Connes, *Noncommutative Geometry*, (Academic Press, 1994).

[2] U. Carow-Watamura et al, Zeit. f. Phys. **C48** (1990) 159.

[3] W.B. Schmidke, J. Wess and B. Zumino, Zeit. f. Phys. **C52** (1991) 471.

[4] S. Majid, J. Math. Phys. **32** (1991) 3246.

[5] S. Majid, J. Math. Phys. **35** (1994) 5025.

[6] V.K. Dobrev, Phys. Lett. **341B** (1994) 133 & **346B** (1995) 427.

[7] V.K. Dobrev, J. Phys. **A28** (1995) 7135.

[8] V.K. Dobrev, *Representations of Quantum Groups and q-Deformed Invariant Wave Equations*, Dr. Habil. Thesis, Tech. Univ. Clausthal 1994, (Papierflieger Verlag, Clausthal-Zellerfeld, 1995, ISBN 3-930697-59-9; V.K. Dobrev, q - difference intertwining operators and q-conformal invariant equations, Acta Appl. Math. **44** (1996) 81-116.

[9] A.O. Barut and R. Rączka, *Theory of Group Representations and Applications*, II edition, (Polish Sci. Publ., Warsaw, 1980).

[10] V.K. Dobrev and B.S. Kostadinov, Phys. Lett. **439B** (1998) 337.

[11] U. Meyer, Comm. Math. Phys. **174** (1995) 447.

[12] V.K. Dobrev, R.I. Gushterski, S.T. Petrov and B.S. Zlatev, Invited talk by V.K.D. at III International Workshop on "Classical and Quantum Integrable Systems" (Yerevan, 1998), Proceedings, eds. L.G. Mardoyan, G.S. Pogosyan and A.N. Sissakian, (Publishing Department of JINR, Dubna, 1998) p. 39.

[13] V.K. Dobrev and B.S. Zlatev, Invited talk by V.K.D. at the VIII Colloquium "Quantum Groups and Integrable Systems", (Prague, 1999), Proceedings: Czech. J. Phys. **50** (2000) 53.

[14] V.K. Dobrev, S.T. Petrov and B.S. Zlatev, Talk by V.K.D. at PDE2000 Conference (Clausthal, Germany), Proc. "Partial Differential Equations and Spectral Theory", ed. M. Demuth et al, *Operator Theory: Advances and Applications, Vol. 126* (Birkhäuser, Berlin, 2001) p. 113.

[15] V.K. Dobrev, S.T. Petrov and B.S. Zlatev, Talk by V.K.D. the 23st International Colloquium on Group Theoretical Methods in Physics, (Dubna, 2000), Proceedings: Physics of Atomic Nuclei, **64** (2001) 2110.

17

[16] M. Jimbo, Lett. Math. Phys. **10** (1985) 63; Lett. Math. Phys. **11** (1986) 247.

[17] V.K. Dobrev, J. Phys. **A27** (1994) 4841 & 6633.

[18] G. Gaspar and M. Rahman, *Basic Hypergeometric Series* (Cambridge U. Press, 1990).

[19] A. Connes and J. Lott, Nucl. Phys. (Proc. Suppl.) **B18** (1991) 29.

[20] V.K. Dobrev, J. Math. Phys. **26** (1985) 235;   V.K. Dobrev, Rep. Math. Phys. **25** (1988) 159.